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The Semilinear Biharmonic Problem with Fully Nonlinear Term

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ABSTRACT: We get one theorem that there exist solutions for the fourth order semilinear elliptic Dirichlet boundary value problem with fully nonlinear term. We prove this result by the critical point theory and the variation of linking method.

Key Words and Phrases: Fourth order elliptic boundary value problem, fully nonlinear term, critical point theory, variation of linking method.

AMS 2000 Mathematics Subject Classifications: 35J20, 35J25

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and let $b \in R$ be a constant. Let $\lambda_k(k = 1, 2, \cdots)$ denote the eigenvalues and $\phi_k(k = 1, 2, \cdots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω with $u = 0$ on $\partial\Omega$, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

We investigate the existence of the nontrivial solutions for the following fourth order semilinear elliptic equation with fully nonlinear term

$$\Delta^2 u + c\Delta u + bu^+ = (u^+)^2 - (u^-)^3 \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where $c \in R$ and $u^+ = \max\{u, 0\}$.

Jung and Choi [5] investigated, by a linking argument, the existence and the multiplicity of the solutions for the following fourth order semilinear elliptic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = b((u + 1)^+ - 1) \qquad \text{in } \Omega, \tag{1.2}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where $c \in R$ and $u^+ = \max\{u, 0\}$.

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Tarantello [8] studied problem (1.2) when $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$. She showed that (1.2) has at least two solutions, one of which is a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [7] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$, then (1.2) has at least three solutions by the Leray-Schauder degree theory. Choi and Jung [2] showed that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.3}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has at least two nontrivial solutions when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and, $s < 0$ or when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. The authors obtained these results by using the variational reduction method. The authors [4] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, (1.2) has at least three nontrivial solutions by using degree theory.

The eigenvalue problem $\Delta^2 u + c\Delta u = \mu u$ in Ω with $u = 0, \qquad \Delta u = 0$ on $\partial\Omega$ has also infinitely many eigenvalues $\mu_k = \lambda_k(\lambda_k - c)$, $k \geq 1$ and corresponding eigenfunctions ϕ_k, $k \geq 1$. We note that $\lambda_1(\lambda_1 - c) < \lambda_2(\lambda_2 - c) \leq \lambda_3(\lambda_3 - c) < \cdots$.

We suppose that $\lambda_1 < \lambda_2 < \lambda_3 \ldots \to +\infty$, and that $\lambda_2 < c < \lambda_3$. Then

$$\lambda_1(\lambda_1 - c) < \lambda_2(\lambda_2 - c) < 0 < \lambda_3(\lambda_3 - c) < \cdots .$$

Jung and Choi [5] showed that: (i) Let $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c) < 0$, $b < \lambda_{k+1}(\lambda_{k+1} - c)$. Then (1.2) has a unique solution.

(ii) Let $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c) < 0 < \lambda_{k+1}(\lambda_{k+1} - c) < \cdots < \lambda_{k+n}(\lambda_{k+n} - c) < b < \lambda_{k+n+1}(\lambda_{k+n+1} - c)$, $k \geq 1$, $n \geq 1$. Then (1.2) has at least two nontrivial solutions.

In section 2, we introduce the Hilbert space and prove $(P.S.)^*_\gamma$- condition for the energy functional. In section 3, we state the main theorem and prove it by using the critical point theory and variation of linking method.

2. Eigenspace and $(P.S.)^*_\gamma$- condition

Let H be a subspace of $L^2(\Omega)$ defined by

$$H = \{u \in L^2(\Omega) | \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty\},$$

where $u = \sum h_k \phi_k \in L^2(\Omega)$ with $\sum h_k^2 < \infty$. Then this is a complete normed space with a norm

$$\|u\| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k(\lambda_k - c) \to +\infty$ and c is fixed, we have

(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.

(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.

(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$.

For the proof of the above results we refer [1].

LEMMA 2.1. *Assume that c is not an eigenvalue of $-\Delta$, $b \neq \lambda_k(\lambda_k - c)$. If $u \in L^2(\Omega)$ and $(u^+)^2 - (u^-)^3 \in L^2(\Omega)$, then all solutions of*

$$\Delta^2 u + c\Delta u + bu^+ = (u^+)^2 - (u^-)^3 \qquad in \quad L^2(\Omega)$$

belong to H.

Proof. Let $u \in L^2(\Omega)$ and $(u^+)^2 - (u^-)^3 \in L^2(\Omega)$. Then $bu^+ \in L^2(\Omega)$ and we put $-bu^+ + (u^+)^2 - (u^-)^3 = \sum h_k\phi_k \in L^2(\Omega)$.

$$u = (\Delta^2 + c\Delta)^{-1}(-bu^+ + (u^+)^2 - (u^-)^3) = \sum \frac{1}{\lambda_k(\lambda_k - c)} h_k\phi_k \in L^2(\Omega).$$

$$\|u\| = \sum |\lambda_k(\lambda_k - c)| \frac{1}{(\lambda_k(\lambda_k - c))^2} h_k^2 \leq C \sum h_k^2 = C\|u\|^2_{L^2(\omega)} < \infty$$

for some $C > 0$. Thus $u \in H$. ∎

With the aid of Lemma 2.1 it is enough that we investigate the existence of the solutions of (1.1) in the subspace H of $L^2(\Omega)$.

Assume that $k \geq 1$ and $\lambda_k < c < \lambda_{k+1}$. We denote by $(\Lambda_i^-)_{i\geq 1}$ the sequence of the negative eigenvalues of $\Delta^2 + c\Delta$, by $(\Lambda_i^+)_{i\geq 1}$ the sequence of the positive ones, so that

$$\Lambda_k^- = \lambda_1(\lambda_1 - c) < \cdots < \Lambda_1^- = \lambda_k(\lambda_k - c) < 0$$

$$< \Lambda_1^+ = \lambda_{k+1}(\lambda_{k+1} - c) < \Lambda_2^+ = \lambda_{k+2}(\lambda_{k+2} - c) < \cdots .$$

We consider an orthonormal system of eigenfunctions $\{e_i^-, e_i^+, i \geq 1\}$ associated with the eigenvalues $\{\Lambda_i^-, \Lambda_i^+, i \geq 1\}$. We set

$$H^+ = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \geq 0\},$$

$$H^- = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \leq 0\}.$$

We define the linear projections $P^- : H \to H^-, P^+ : H \to H^+$.

We also introduce two linear operators $R : H \to H^+, S : H \to H^-$ by

$$S(u) = \sum_{i=1}^{\infty} \frac{a_i^- e_i^-}{\sqrt{-\Lambda_i^-}}, R(u) = \sum_{i=1}^{\infty} \frac{a_i^+ e_i^+}{\sqrt{\Lambda_i^+}}$$

if

$$u = \sum_{i=1}^{\infty} a_i^- e_i^- + \sum_{i=1}^{\infty} a_i^+ e_i^+.$$

It is clear that S and R are compact and self adjoint on H.

Definition 2.1. Let $I_b : H \to R$ be defined by

$$I_b(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|[Au]^+\|^2 - \int_\Omega F(Au)dx$$

where $A = R + S$ and $F(s) = \int_0^s f(x,\tau)d\tau,\ f(x,\tau) = (\tau^+)^2 - (\tau^-)^3$.

It is straightforward that

$$\nabla I_b(u) = P^+u - P^-u + bA(Au)^+ - Af(Au).$$

Following the idea of Hofer [3] one can show that

Proposition 2.2. *$I_b \in C^{1,1}(H,R)$. Moreover $\nabla I_b(u) = 0$ if and only if $w = (R+S)(u)$ is a weak solution of (P), that is,*

$$\int_\Omega (w(v_{tt} + v_{xxxx}) + b[w]^+v)dxdt = \int_\Omega f(w)vdxdt \text{ for all smooth } v \in H.$$

In this section, we suppose $b > 0$. Under this assumption, we have a concern with multiplicity of solutions of equation (1.1). Here we suppose that f is defined by equation $f(x,\tau) = (\tau^+)^2 - (\tau^-)^3$.

In the following, we consider the following sequence of subspaces of $L^2(R^N)$:

$$H_n = (\oplus_{i=1}^n H_{\Lambda_i^-}) \oplus (\oplus_{i=1}^n H_{\Lambda_i^+})$$

where H_Λ is the eigenspace associated to Λ and $H_{\Lambda_i^-} = \phi$ if $i > k$.

Lemma 2.5. *The functional I_b satisfies $(P.S.)_\gamma^*$ condition, with respect to (H_n), for all γ.*

Proof. Let (k_n) be any sequence in N with $k_n \to \infty$. And let (u_n) be any sequence in H such that $u_n \in H_n$ for all n, $I_b(u_n) \to \gamma$ and $\nabla(I_b)\mid_{H_{k_n}}(u_n) \to 0$.

First, we prove that (u_n) is bounded. By contradiction let $t_n = \|u_n\| \to \infty$ and set $\hat{u_n} = u_n/t_n$. Up to a subsequence $\hat{u_n} \rightharpoonup \hat{u}$ in H for some $\hat{u}$ in H. Moreover

$$\begin{aligned} 0 \ \leftarrow\ & < \nabla(I_b)_{H_{k_n}}(u_n), \hat{u_n} > -\frac{2}{t_n}I_b(u_n) \\ =\ & \frac{2}{t_n}\int_\Omega F(Au_n)dx - \frac{1}{t_n}\int_\Omega f(Au_n)Au_n dx \\ =\ & \int_\Omega -\frac{1}{3}(t_n)^{p-1}[(A\hat{u_n})^+]^p + \frac{6}{4}(t_n)^{q-1}[(A\hat{u_n})^-]^q dx. \end{aligned}$$

Since $t_n \to \infty$, $(A\hat{u_n})^+ \to 0$ and $(A\hat{u_n})^- \to 0$. This implies $A\hat{u} = 0$ and $\hat{u} = 0$, a contradiction.

So (u_n) is bounded and we can suppose $u_n \rightharpoonup u$ for some $u \in H$. We know that

$$\nabla(I_b)_{H_{k_n}}(u_n) = P^+u_n - P^-u_n + bA(Au_n)^+ - Af(Au_n).$$

Since A is the compact operator, $P^+u_n - P^-u_n$ converges strongly, hence $u_n \to u$ strongly and $\nabla I_b(u) = 0$. ∎

3. An Application of Linking Theory

Fixed Λ_i^- and $\Lambda_i^- < -b < \Lambda_{i-1}^-$. We prove the Theorem via a linking argument.

First of all, we introduce a suitable splitting of the space H. Let

$$Z_1 = \oplus_{j=i+1}^{\infty} H_{\Lambda_j^-}, Z_2 = H_{\Lambda_i^-}, Z_3 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-} \oplus H^+,$$

where $H_{\Lambda_j^-} = \phi$ if $j > k$.

Lemma 3.1. *There exists R such that $\sup_{v\in Z_1\oplus Z_2, \|v\|=R} I_b(v) \le 0$.*

Proof. If $v \in Z_1 \oplus Z_2$ then

$$I_b(v) = -\frac{1}{2}\|v\|^2 + \frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx.$$

Since

$$\frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx = \int_\Omega \frac{b}{2}([Sv]^+)^2 - \frac{1}{3}([Sv]^+)^3 - \frac{1}{4}([Sv]^-)^4 dx,$$

there exists R such that $\frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx \le 0$ for all $\|v\| = R$. Hence

$$I_b(v) \le -\frac{1}{2}\|v\|^2 \le 0$$

∎

Lemma 3.2. *There exists ρ such that $\inf_{u\in Z_2\oplus Z_3, \|u\|=\rho} I_b(u) > 0$.*

Proof. Let $\sigma \in [0,1]$. We consider the functional $I_{b,\sigma} : Z_2 \oplus Z_3 \to R$ defined by

$$I_{b,\sigma}(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|[Au]^+\|^2 - \sigma \int_\Omega F(Au)dx.$$

We claim that there exists a ball $B_\rho = \{u \in Z_2 \oplus Z_3 | \|u\| < \rho\}$ such that

(1) $I_{b,\sigma}$ are continuous with respect to σ,

(2) $I_{b,\sigma}$ satisfies $(P.S)$ condition,

(3) 0 is a minimum for $I_{b,0}$ in B_ρ,

(4) 0 is the unique critical point of $I_{b,\sigma}$ in B_ρ.

Then by a continuation argument of Li-Szulkin's [6], it can be shown that 0 is a local minimum for $I_b|_{Z_2\oplus Z_3} = I_{b,1}$ and Lemma is proved.

The continuity in σ and the fact that 0 is a local minimum for $I_{b,0}$ are straightforward. To prove $(P.S.)$ condition one can argue as in the previous Lemma, when dealing with I_b.

To prove that 0 is isolated we argue by contradiction and suppose that there exists a sequence (σ_n) in $[0,1]$ and sequence (u_n) in $Z_2 \oplus Z_3$ such that $\nabla I_{b,\sigma_n}(u_n) = 0$ for all $n, u_n \neq 0$, and$u_n \to 0$. Set $t_n = \|u_n\|$ and $\hat{u_n} = u_n/t_n$ then $t_n \to 0$. Let $\hat{v_n} = P^- \hat{u_n}$ and $\hat{w_n} = P^+ \hat{u_n}$. Since $\hat{v_n}$ varies in a finite dimensional space, we can suppose that $\hat{v_n} \to \hat{v}$ for some $\hat{v}$. We get

$$\frac{1}{t_n}\nabla I_{b,\sigma}(u_n) = \hat{w_n} - \hat{v_n} + \frac{b}{t_n}A(Au_n)^+ - \frac{\sigma_n}{t_n}Af(Au_n) = 0. \tag{1}$$

Multiplying by $\hat{w_n}$ yields

$$\|\hat{w_n}\|^2 = \frac{\sigma_n}{t_n}\int_\Omega f(Au_n)A\hat{w_n}dx - \frac{b}{t_n}\int_\Omega (Au_n)^+ A\hat{w_n}dx.$$

We know that

$$\begin{aligned}\int_\Omega (Au_n)^+ A\hat{w_n}dx &= \int_\Omega P^+(Au_n)^+ A\hat{u_n}dx \\ &= \int_\Omega P^+(Au_n)^+ (A\hat{u_n})^+ dx.\end{aligned}$$

Since $b > 0$, there exists a sequence (ϵ_n) such that $\epsilon_n \to 0$ and $0 < \epsilon_n < b$ for all n. That is

$$\frac{b}{t_n}\int_\Omega (Au_n)^+ A\hat{w_n}dx \geq \frac{\epsilon_n}{t_n}\int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dx.$$

Then

$$\begin{aligned}\|\hat{w_n}\|^2 &\leq \frac{1}{t_n}\int_\Omega f(Au_n)A\hat{w_n}dx - \frac{\epsilon_n}{t_n}\int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dx \\ &\leq \int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{w_n}|dx + \epsilon_n\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dx.\end{aligned}$$

Since A is a compact operator

$$\begin{aligned}|f(Au_n)| &= |\{([t_nA\hat{u_n}]^+)^{p-1} - ([t_nA\hat{u_n}]^-)^{q-1}\}| \\ &\leq {t_n}^{p-1}|[A\hat{u_n}]^+|^{p-1} + {t_n}^{q-1}|[A\hat{u_n}]^-|^{q-1} \\ &\leq {t_n}^m(M_1 + {t_n}^{M-m}M_2)\end{aligned}$$

for some M_1 and M_2 where $m = \min\{p-1, q-1\}$ and $M = \max\{p-1, q-1\}$. We get that

$$\int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{w_n}|dx \leq {t_n}^m(M_1 + {t_n}^{M-m}M_2)\int_\Omega |A\hat{w_n}|dx \leq o(1).$$

Hence

$$\|\hat{w_n}\|^2 \leq o(1) + \epsilon_n\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dx. \tag{2}$$

Since $\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dx$ is bounded and equation (7) holds for every ϵ_n, $\hat{w_n} \to 0$ and so $(\hat{u_n})$ converges. Since $|f(Au_n)| \leq {t_n}^m(M_1 + {t_n}^{M-m}M_2)$, we get

$$\frac{\sigma_n}{t_n}|f(Au_n)| \leq \frac{1}{t_n}|f(Au_n)| \leq {t_n}^{m-1}(|M_1 + {t_n}^{M-m}M_2) \leq o(1).$$

Then $\frac{\sigma_n}{t_n}Af(Au_n) \to 0$. From equation (6), $(\hat{v_n})$ converges to zero, but this is impossible if $\|\hat{(u_n)}\| = 1$. ∎

We give the definitions for the next step:

Definition 3.3. Let H be an Hilbert space, $Y \subset H$, $\rho > 0$ and $e \in H \setminus Y$, $e \neq 0$. Set:

$$\begin{aligned}
B_\rho(Y) &= \{x \in Y \mid \|x\| \leq \rho\}, \\
S_\rho(Y) &= \{x \in Y \mid \|x\| = \rho\}, \\
\triangle_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| \leq \rho\}, \\
\Sigma_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| = \rho\} \cup \{v \mid v \in Y, \|v\| \leq \rho\}.
\end{aligned}$$

Theorem 3.4. *If $\Lambda_i^- \leq -b (i = 1, 2, \cdots, k)$ then problem (1.1) has at least one nontrivial solution.*

Proof. Let $e \in Z_2$. By Lemma 3.1 and Lemma 3.2, for a suitable large R and a suitable small ρ, we have the linking inequality

$$\sup I_b(\Sigma_R(e, Z_1)) < \inf I_b(S_\rho(Z_2 \oplus Z_3)). \tag{3}$$

Moreover $(P.S.)^*_\gamma$ holds. By standard linking arguments, it follows that there exists a critical point u for I_b with $\alpha \leq I_b(u) \leq \beta$, where $\alpha = \inf I_b(S_\rho(Z_2 \oplus Z_3))$ and $\beta = \sup I_b(\Delta_R(e, Z_1))$. Since $\alpha > 0$, then $u \neq 0$. ∎

We assume in this section that $i \geq 2$ and we set

$$W_1 = \oplus_{j=i}^\infty H_{\Lambda_j^-}, W_2 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-}, W_3 = H^+.$$

Notice that $W_1 = Z_1 \oplus Z_2$ and $W_2 \oplus W_3 = Z_3$.

Lemma 3.5. $\liminf_{\|u\| \to +\infty, u \in W_1 \oplus W_2} I_b(u) \leq 0.$

Proof. Let $(u_n)_n$ be a sequence in $W_1 \oplus W_2$ such that $\|u_n\| \to \infty$. We set $t_n = \|u_n\|$ and $\hat{u_n} = u_n/t_n$. Since S is a compact operator,

$$\begin{aligned}\frac{b}{2}\frac{\|[Su_n]^+\|^2}{t_n^2} &- \int_\Omega \frac{F(Su_n)}{t_n^2}dx \\ &= \int_\Omega \frac{b}{2}([S\hat{u_n}]^+)^2 - \frac{{t_n}^{p-2}}{p}([S\hat{u_n}]^+)^p - \frac{{t_n}^{q-2}}{q}([S\hat{u_n}]^-)^q dx \\ &\to -\infty.\end{aligned}$$

Then

$$\frac{I_b(u_n)}{\|u_n\|^2} = -\frac{1}{2} + \frac{b}{2}\frac{\|[Su_n]^+\|^2}{t_n^2} - \int_\Omega \frac{F(Su_n)}{t_n^2}dx \to -\infty.$$

Hence

$$\liminf_{\|u\| \to +\infty, u \in W_1 \oplus W_2} I_b(u) \leq 0.$$

■

LEMMA 3.6. *There exists $\hat{\rho}$ such that* $\inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) > 0$.

Proof. Repeating the same arguments used in Lemma 3.2, we get the conclusion. ■

THEOREM 3.7. *Let $i \geq 2$. If $\Lambda_i^- \leq -b$ then problem (1) has at least two nontrivial solution.*

Proof. Using the conclusion of Theorem 3.4, we have that there exist a nontrivial critical point u with

$$I_b(u) \leq \sup I_b(\Delta_R(e, Z_1))$$

where e, R were given in Lemma 3.1 and 3.2. We can choose that $\hat{R} \geq R$. Take any $\hat{e}$ in W_2, then we have a second linking inequality,

$$\sup I_b(\Sigma_{\hat{R}}(\hat{e}, W_1)) \leq \inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)).$$

Since $(P.S.)_\gamma^*$ holds, there exists a critical point $\hat{u}$ such that

$$\inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) \leq I_b(\hat{u}) \leq \sup I_b(\Delta_{\hat{R}}(\hat{e}, W_1)).$$

Since $\hat{R} \geq R$ and $Z_1 \oplus Z_2 = W_1$,

$$\Delta_R(e, Z_1) \subset B_{\hat{R}}(W_1) \subset \Sigma_{\hat{R}}(\hat{e}, W_1).$$

Then

$$\begin{aligned} I_b(u) &\leq \sup I_b(\Delta_R(e, Z_1)) \\ &\leq \sup I_b(\Sigma_{\hat{R}}(\hat{e}, W_1)) < \inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) \leq I_b(\hat{u}).\end{aligned}$$

Hence $u \neq \hat{u}$. ∎

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